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Approximate boundary controllability of Sobolev-type stochastic differential systems ☆

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Abstract The objective of this paper is to investigate the approximate boundary controllability of Sobolev-type stochastic differential systems in Hilbert spaces. The control function for this system is suitably constructed by using the infinite dimensional controllability operator. Sufficient conditions for approximate boundary controllability of the proposed problem in Hilbert space is established by using contraction mapping principle and stochastic analysis techniques. The obtained results are extended to stochastic differential systems with Poisson jumps. Finally, an example is provided which illustrates the main results.

MATHEMATICS SUBJECT CLASSIFICATION: 34K50; 60H10; 93B05; 93E03

1. Introduction

In many cases, the accurate analysis, design, and assessment of systems subjected to realistic environments must take into account the potential of random loads and randomness in the system properties. Randomness is intrinsic to the mathematical formulation of many phenomena such as fluctuations in the stock market, or noise in communication networks. To build more realistic models in economics, social sciences, chemistry, finance, physics and other areas, stochastic effects need to be taken into account. Mathematical modeling of such systems often leads to differential equations with random parameters. The use of deterministic equations that ignore the randomness of the parameter or replace them by their mean values can result in gross errors. All such problems are mathematically modeled and described by various stochastic systems described by stochastic differential equations, stochastic delay equations, and in some cases stochastic integro-differential equations which are mathematical models for phenomena with irregular fluctuations. Stochastic differential equations are important from the viewpoint of applications since they incorporate (natural) randomness into the mathematical description of phenomena, thereby describing it more accurately. The theory of stochastic differential systems has become an important area of investigation in the past two decades because of their applications to various problems arising in communications, control technology, mechanics,

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electrical engineering, medicine, biology, aviation, spaceflight, material, robot, bioengineering, etc. [1,2]. This is due to the fact that most problems in a real life situation to which mathematical models are applicable are basically stochastic rather than deterministic (see [3]).

Mathematical control theory is one of the important concept in the study of steering the dynamical system from given initial state to any other final state or to neighborhood of the final state under some admissible control input. The controllability problem for an evolution equation is also consist of driving the solution of the system to a prescribed final target state (exactly or in some approximate way) in a finite interval of time (see [4] and references therein). Problems of this type are common in science and engineering and, in particular, they arise often in the context of flow control, the control of flexible structures appearing in flexible robots and large space structures, quantum chemistry, etc. (see [5]). From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. It is obvious that exact controllability is an essentially stronger notion than approximate controllability. Exact controllability always implies approximate controllability. The converse statement is generally false. However, it should be addressed that in the case of finite dimensional systems, the notions of exact and approximate controllability coincide. Controllability results for a class of fractional-order neutral evolution systems was discussed in [6]. Sakthivel et al, [7] investigated the problem of approximate controllability for a class of nonlinear impulsive differential equations with state-dependent delay by using semigroup theory and fixed point technique. In recent years, controllability problems for various types of deterministic and stochastic dynamical system have been studied in different directions (see [8–13] and references therein). In the literature, there are different definitions of controllability for SDEs, both for linear and nonlinear dynamical systems [8,9,14]. In particular, Klamka [15] derived the stochastic controllability of linear systems with delay in control. Muthukumar et al. [16] proved the approximate controllability of nonlinear stochastic evolution systems with time varying delays with preassigned responses. Sakthivel et al. [17] investigated the approximate controllability of second order stochastic differential equations with impulsive effects by using the Holders inequality, stochastic analysis, and fixed point strategy. Shen et al. [18] proved approximate controllability of abstract stochastic impulsive systems with multiple time-varying delays by using the natural assumptions that the corresponding linear system is approximately controllable. Sakthivel et al. [19,20] studied approximate controllability of fractional stochastic system by using fixed point theorem with stochastic analysis theory.

Especially in the past two decades, applications resulting from technological developments gave rise to the study of infinite dimensional linear systems governed by partial differential equations. In engineering, these systems are referred to as distributed parameter systems. Systems of this type appear for instance in steel making plants, where the heat distribution on a metal slab has to be governed, in biology, where the size of a bacteria population has to be controlled or in electrical engineering, where optimal operation of power plants has to be calculated (see [2]). These examples fit into a class of systems where control cannot be exceeded everywhere. It is for instance only possible to heat the metal slab at the boundary, to control the population size at a certain age or to generate current in the power plants of an electrical network. Several abstract settings have been developed to describe the distributed control systems on a domain in which the control is acted through the boundary. But in these approaches one can encounter the difficulty for the existence of sufficiently regular solution to state space system, the control must be taken in a space of sufficiently smooth functions.

A semigroup approach to boundary input problems for linear differential equations was first presented by Fattorini [21]. This approach was extended by Balakrishnan [22] where he showed that the solution of a parabolic boundary control equation with L_2 controls can be expressed as a mild solution to an operator equation. Barbu [23] investigated a class of boundary distributed linear control systems in Banach spaces. MacCamy et al. [24] obtained the approximate boundary controllability for the heat equations. Han et al. [25] also studied the boundary controllability of differential equations with nonlocal condition by using Banach fixed point theorem. Many authors studied the boundary controllability of differential equations in deterministic cases (see [26–30] and references therein). Balachandran et al. [31] established the sufficient conditions for the boundary controllability of various types of nonlinear Sobolev-type systems including integro differential systems in Banach spaces. A Sobolev-type equation appears in a variety of physical problems such as flow of fluids through fissured rocks, thermodynamics, and propagation of long waves of small amplitude (see [32,33]). Wang [34] addressed the approximate boundary controllability results for semilinear delay differential equations by using the corresponding linear system which is approximately boundary controllable. Li et al. [35] showed that the boundary controllability of nonlinear stochastic differential inclusions by using a fixed point theorem for condensing maps due to Leray-Schauder nonlinear alternative theorem. If the semigroup is compact, then assumptions (H_2) in [35] is valid if and only if the state space is finite dimensional. As a result, the applications are restricted to stochastic ordinary differential control systems. Motivated by [31,34,35], the aim of the proposed work is to obtain the approximate boundary controllability of the following Sobolev-type stochastic differential systems without using the hypothesis (H_2) in [35]

$$\begin{aligned} d(Fx(t)) &= (\rho x(t) + f(t, x(\gamma_1(t)), x(\gamma_2(t)), \ldots, x(\gamma_n(t))))dt \\ &\quad + g(t, x(\gamma_1(t)), x(\gamma_2(t)), \ldots, x(\gamma_n(t)))dW(t), \quad t \in J = [0, b], \\ \tau x(t) &= B_1 u(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where the state variable $x(\cdot)$ takes values in a Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ and the control function $u(\cdot)$, takes values in Hilbert space U. $B_1 : U \to H$ is a linear continuous operator. Let $\mathcal{C} := C(J; L_2(\Omega, H))$ be the space of all real valued measurable continuous functions from J into H. Let $\rho : D(\rho) \subseteq \mathcal{C} \to R(\rho) \subseteq H$ is a closed, densely defined linear operator, where $D(\rho)$ is the domain of ρ and $R(\rho)$ is the range of ρ and $\tau : D(\tau) \subseteq \mathcal{C} \to R(\tau) \subseteq H$ is a linear operator with τ be a partial differential operator acting on the boundary of H. Let K be a another separable Hilbert space. Suppose $\{W(t)\}_{t \geqslant 0}$ is a given K- valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geqslant 0$. We are also employing the same notation $\|\cdot\|$ for the norm of $L(K, H)$, where $L(K, H)$ denotes the space of all bounded operators from K into H, simply $L(H)$ if $K = H$. Let $F : D(F) \subset \mathcal{C} \to R(F) \subset H$ be a linear operator, the nonlinear function f be a

H-valued map defined on $J \times H^n$ and g be a $L_Q(K,H)$ valued map defined on $J \times H^n$. Here, $L_Q(K,H)$ denotes the space of all $Q-$ Hilbert Schmidt operators from K into H. $\gamma_i(t)$: $J \to J$, $i = 1, 2, \ldots, n$ are continuous functions. The initial data x_0 are an $\mathfrak{F}_0$-adapted H-valued random variable independent of Wiener process W. Let $y(t) = F\,x(t)$ then (1) can be written as

$$\begin{aligned} d(y(t)) &= (\rho F^{-1}y(t) + f(t, F^{-1}y(\gamma_1(t)), F^{-1}y(\gamma_2(t)), \ldots, F^{-1}y(\gamma_n(t))))dt \\ &\quad + g(t, F^{-1}y(\gamma_1(t)), F^{-1}y(\gamma_2(t)), \ldots, F^{-1}y(\gamma_n(t)))dW(t), \quad t \in J, \\ \bar{\tau}y(t) &= B_1u(t), \\ y(0) &= y_0, \end{aligned} \tag{2}$$

where $\bar{\tau} = \tau F^{-1}$. Let A:$D(A) \subset H \to H$ be a linear operator defined by $D(AF^{-1}) = \{\alpha \in D(\rho F^{-1}); \bar{\tau}\alpha = 0\}$, $AF^{-1}\alpha = \rho F^{-1}\alpha$ for $\alpha \in D(A\ F^{-1})$.

The operators A:$D(A) \subset H \to H$ and $F : D(F) \subset \mathcal{C} \to R(F) \subset H$ satisfy the following hypotheses [12]

- (S_1) A and F are closed linear operators,
- (S_2) $D(F) \subset D(A)$ and F is bijective,
- (S_3) $F^{-1} \to D(F)$ is continuous,
- (S_4) The resolvent $R(\vartheta, AF^{-1})$ is compact for some $\vartheta \in \rho(AF^{-1})$, the resolvent set of AF^{-1}.

The rest of this paper is organized as follows. Section 2 describes some notations, lemmas and preliminary results of stochastic settings. Sections 3 and 4 are devoted to the study of existence, approximate boundary controllability of Sobolev-type stochastic differential systems, and the boundary controllability results of the system with Poisson jumps. In Section 5, an example is provided to illustrate the application of the main result. Section 6 contains the conclusion.

2. Preliminaries

For more details of this section, the reader may refer [1,3,9,10,14,16–20] and the references therein.

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ algebras $\{\mathfrak{F}_t, t \in J\}$ satisfying $\mathfrak{F}_t \subset \mathfrak{F}$. H-valued random variable is a $\mathfrak{F}$ measurable function $x(t)$:$\Omega \to H$, and a collection of random variable $S = \{x(t, \omega)$:$\Omega \to H|_{t\in J}\}$ is called a stochastic process. Usually, we suppress the dependence on $\omega \in \Omega$and write $x(t)$ instead of $x(t,\omega)$ and $x(t)$:$J \to H$ in the place of S. Let $\beta_n(t)(n = 1, 2, \ldots)$ be a sequence of real valued one dimensional standard Brownian motion mutually independent over $(\Omega, \mathfrak{F}, P)$. Set $W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\zeta_n$, $t \geqslant 0$ where $\lambda_n \geqslant 0$, $n = 1, 2, \ldots$ are nonnegative real numbers and $\{\zeta_n\}$, $n = 1, 2, \ldots$ is complete orthonormal basis in K. Let $Q \in L(K,K)$ be an operator defined by $Q\zeta_n = \lambda_n\zeta_n$ with finite $Tr(Q) = \sum_{n=1}^{\infty}\lambda_n < \infty$ (Tr denotes the Trace of the operator). Then, the above K-valued stochastic process $W(t)$ is called a Q-Wiener process. We assume that $\mathfrak{F}_t = \sigma(W(s) : 0 \leqslant s \leqslant t)$ is the σ- algebra generated by W and $\mathfrak{F}_t = \mathfrak{F}$. Let $\psi \in L(K,H)$ and define

$$\|\psi\|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \| \sqrt{\lambda_n}\psi\zeta_n\|^2.$$

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert Schmidt operators ψ:$K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = \langle \psi, \psi \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly measurable, square integrable H- valued random variables denoted by $L_2(\Omega, \mathfrak{F}, P; H) = L_2(\Omega, H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = \left(E\|x(\cdot; w)\|_H^2\right)^{\frac{1}{2}}$, where the expectation E is defined by $E(\hbar) = \int_\Omega \hbar(w)dP$. Similarly, $L_2^{\mathfrak{F}}(\Omega, H)$ denotes the Banach space of all $\mathfrak{F}_t$- measurable, square integrable random variables, such that $\int_\Omega \|x(t,\cdot)\|_{L_2}^2 dt < \infty$. $C(J, L_2(\Omega,H))$ is the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the condition $\sup_{t\in J}E\|x(t)\|^2 < \infty$.

Let $y(t)$ be the solution of (2). Then defined the function $z(t) = y(t) - Bu(t)$. From the assumptions, it follows that $z(t) \in D(AF^{-1})$. Hence, (2) can be written as

$$\begin{aligned} d(z(t)) &= (AF^{-1}z(t) + \rho F^{-1}Bu(t) - Bu'(t))dt + f(t, F^{-1}y(\gamma_1(t)), F^{-1}y(\gamma_2(t)), \ldots, F^{-1}y(\gamma_n(t)))dt \\ &\quad + g(t, F^{-1}y(\gamma_1(t)), F^{-1}y(\gamma_2(t)), \ldots, F^{-1}y(\gamma_n(t)))dW(t), \\ z(0) &= y(0) - Bu(0), \end{aligned}$$

and the mild solution of (2) is given by [31,35]

$$\begin{aligned} y(t) &= T(t)y(0) + \int_0^t [T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu(s)ds \\ &\quad + \int_0^t T(t-s)f(s, F^{-1}y(\gamma_1(s)), F^{-1}y(\gamma_2(s)), \ldots, F^{-1}y(\gamma_n(s)))ds \\ &\quad + \int_0^t T(t-s)g(s, F^{-1}y(\gamma_1(s)), F^{-1}y(\gamma_2(s)), \ldots, F^{-1}y(\gamma_n(s)))dW(s), \end{aligned}$$

which is well defined. Hence, the mild solution of the system (1) is given by

$$\begin{aligned} x(t) &= F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu(s)ds \\ &\quad + \int_0^t F^{-1}T(t-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\ &\quad + \int_0^t F^{-1}T(t-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s), \end{aligned} \tag{3}$$

To prove the main results, we assume the following hypotheses [31,36].

- (H_1) $D(\rho) \subset D(\tau)$ and the restriction of τ to $D(\rho)$ is continuous relative to graph norm of $D(\rho)$.
- (H_2) The operator AF^{-1} is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ such that $\|T(t)\|^2 \leqslant M$ for some $M \geqslant 1$.
- (H_3) There exists a linear continuous operator B: $U \to Z$ such that $\rho F^{-1}B \in L(U,Z)$, $\bar{\tau}(Bu) = B_1u$ for all $u \in U$. Also, $\|Bu\| \leqslant c_0\|B_1u\|$ for all $u \in U$, where c_0 is a constant.
- (H_4) For all $t \in (0, b]$ and $u \in U$, $T(t)Bu \in D(AF^{-1})$ and $AF^{-1}T(t)B$ is a linear operator. Moreover, there exists a positive function $\kappa \in L_1(0, b)$ such that $\|AF^{-1}T(t)$-$B\|^2 \leqslant \kappa(t)$ a.e $t \in (0, b)$.
- (H_5) There exist constants $N_1, N_2 > 0$ such that $\int_0^b \kappa(t)dt \leqslant N_1$ and $|\ F^{-1}|\ ^2 \leqslant N_2$.
- (H_6) The functions f:$J \times H^n \to H$ and g:$J \times H^n \to L_Q(K,H)$ are continuous and there exists constants C_1, C_2, for $t \in J$ and $x_1(\gamma_i(s))$, $x_2(\gamma_i(s)) \in H$, $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} &\|f(t, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) - f(t, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s)))\|^2 \\ &\quad + \|g(t, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) - g(t, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s)))\|_Q^2 \\ &\leqslant C_1 \sum_{i=1}^{n} \|x_1(\gamma_i(s)) - x_2(\gamma_i(s))\|^2, \\ &C_2 = \max_{t\in J}(\|f(t, 0, \ldots, 0)\|^2 + \|g(t, 0, \ldots, 0)\|^2) \end{aligned}$$

- (H_7) There exists a constant C_3 such that for every x_1, $x_2 \in H$

$$\|x_1(\gamma_i(t)) - x_2(\gamma_i(t))\|^2 \leqslant C_3\|x_1(t) - x_2(t)\|^2, \quad \text{for} \quad i = 1, 2, \ldots, n$$

(H_8) For each $0 \leqslant t < b$, the operator $\lambda R(\lambda, \Gamma_t^b) = \lambda(\lambda I + \Gamma_t^b)^{-1}$ converges to zero in the strong operator topology as $\lambda \to 0^+$, where the controllability Gramian Γ_t^b, associated with (1), is defined as [8,9,18–20]

$$\Gamma_t^b = \int_t^b F^{-1}[T(b-s)\rho F^{-1} - AF^{-1}T(b - s)]BB^{\star}F^{-1}[T(b-s)\rho F^{-1} - AF^{-1}T(b-s)]^{\star} ds.$$

Definition 2.1. The system (1) is said to be approximately boundary controllable on [0,*b*] if $\overline{\mathcal{R}(b; x_0, u)} = L_2(\Omega, \mathfrak{F}_b, H)$, where the reachable set $\mathcal{R}(b; x_0, u)$ is defined as $\mathcal{R}(b; x_0, u) = \{x(b; x_0, u), u(\cdot) \in L_2^{\mathfrak{F}}(J, U)\}$. Here $x(b; x_0, u)$ is called the system state at time $t = b$ corresponding to the initial condition x_0 and the control input u.

Lemma 2.1 [9]. *For any* $x_b \in L_2(\Omega, \mathfrak{F}_b, H)$, *there exists* $\varphi \in L_2^{\mathfrak{F}}(\Omega, L_2(0, b; L_Q(K, H)))$ *such that* $x_b = Ex_b + \int_0^b \varphi(s)dW(s)$.

To obtain the approximate controllability result, for any $x_b \in L_2(\Omega, \mathfrak{F}_b, H)$, by selecting proper control u^{λ}(for any given $\lambda \in (0, 1]$), there exists a mild solution $x^{\lambda}(\cdot, x_0, u^{\lambda}) \in \mathcal{C}$ for system (1), and then we prove that $x^{\lambda} \to x_b$ in H as $\lambda \to 0^+$, which reaches the result. For all $\lambda > 0$, define the control for the system (1) as

$$\begin{aligned} u^{\lambda}(t,x) = {} & B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star}\left(\lambda I + \Gamma_0^b\right)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) \\ & - B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star}\int_0^t \left(\lambda I + \Gamma_s^b\right)^{-1}F^{-1}T(b-s) \\ & \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\ & - B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star} \\ & \times \int_0^t \left(\lambda I + \Gamma_s^b\right)^{-1}F^{-1}T(b-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s) \\ & + B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star}\int_0^t \left(\lambda I + \Gamma_s^b\right)^{-1}\varphi(s)dW(s). \end{aligned} \tag{4}$$

Using this control function, we define the operator Φ^{λ} on $\mathcal{C}$ as follows

$$\begin{aligned} (\Phi^{\lambda}x)(t) = {} & F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu^{\lambda}(s,x)ds \\ & + \int_0^t F^{-1}T(t-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\ & + \int_0^t F^{-1}T(t-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s), \quad t \in J. \end{aligned} \tag{5}$$

3. Main results

3.1. Existence of solutions

Taking into account the above notations, definitions and lemmas, we shall derive the existence of solution for the nonlinear stochastic system (1) by using the contraction mapping principle. The existence of solutions to system (1) is a natural premise to carry out the study of boundary controllability.

Theorem 3.1. *Suppose the hypotheses* (H_1)*–*(H_7) *hold. If*

$$3\left(\frac{\mathcal{N}}{\lambda^2}N_2(bM\|\rho F^{-1}\|^2\|B\|^2 + N_1) + N_2M(b + Tr(Q))C_1nC_3\right)b < 1, \tag{6}$$

then the operator Φ^{λ} *has a fixed point in* $\mathcal{C}$.

Proof. We prove the existence of a fixed point of the operator Φ^{λ} by using the contraction mapping theorem. Initially, we show that $\Phi^{\lambda} : \mathcal{C} \to \mathcal{C}$. We shall first study the control function $u^{\lambda}(t,x)$. Let $x_1, x_2 \in \mathcal{C}$. From the Holder's inequality and the assumption on the data, we obtain

$$\begin{aligned} & E\|u^{\lambda}(t,x_1) - u^{\lambda}(t,x_2)\|^2 \\ & = E\|B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star}\int_0^t \left(\lambda I + \Gamma_s^b\right)^{-1}F^{-1}T(b-s) \\ & \quad \times (f(s, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) - f(s, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s))))ds \\ & \quad + B^{\star}F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^{\star}\int_0^t \left(\lambda I + \Gamma_s^b\right)^{-1}F^{-1}T(b-s) \\ & \quad \times (g(s, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) - g(s, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s))))dW(s)\|^2, \\ & \leqslant 2\frac{N_2^2}{\lambda^2}(M\|\rho F^{-1}\|^2\|B^{\star}\|^2 + \kappa(t))M(b + Tr(Q))C_1\int_0^t \Big(E\|x_1(\gamma_1(s)) - x_2(\gamma_1(s))\|^2 \\ & \quad + E\|x_1(\gamma_2(s)) - x_2(\gamma_2(s))\|^2 + \ldots + E\|x_1(\gamma_n(s)) - x_2(\gamma_n(s))\|^2\Big)ds, \\ & \leqslant 2\frac{N_2^2}{\lambda^2}(M\|\rho F^{-1}\|^2\|B^{\star}\|^2 + \kappa(t))M(b + Tr(Q))C_1nC_3\int_0^t E\|x_1(s) - x_2(s)\|^2ds, \\ & \leqslant \frac{\mathcal{N}}{\lambda^2}\int_0^t E\|x_1(s) - x_2(s)\|^2ds, \end{aligned}$$

Similarly

$$\begin{aligned} E\|u^{\lambda}(t,x)\|^2 & \leqslant 4\frac{N_2}{\lambda^2}(M\|\rho F^{-1}\|^2\|B^{\star}\|^2 + \kappa(t))\Big(\|x_b\|^2 + N_2M|Fx(0)|^2 + N_2M(b + Tr(Q)) \\ & \quad \times \Big(C_2b + C_1nC_3\int_0^t E\|x(s)\|^2ds\Big)\Big), \\ & \leqslant \frac{\widehat{\mathcal{N}}}{\lambda^2}\Big(C_2b + C_1nC_3\int_0^t E\|x(s)\|^2ds\Big), \end{aligned}$$

where $\mathcal{N} = 2N_2^2(M\|\rho F^{-1}\|^2\|B^{\star}\|^2 + \kappa(t))M(b + Tr(Q))C_1nC_3$ and $\widehat{\mathcal{N}} = 4N_2(M\|\rho F^{-1}\|^2\|B^{\star}\|^2 + \kappa(t))(\|x_b\|^2 + N_2M|Fx(0)|^2 + N_2M(b + Tr(Q))$. Now consider

$$\begin{aligned} & E\|(\Phi^{\lambda}x)(t)\|^2 \\ & = E\|F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu^{\lambda}(s,x)ds \\ & \quad + \int_0^t F^{-1}T(t-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\ & \quad + \int_0^t F^{-1}T(t-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s)\|^2, \\ & \leqslant 4N_2M|Fx(0)|^2 + 8N_2\frac{\widehat{\mathcal{N}}}{\lambda^2}\int_0^t (M\|\rho F^{-1}\|^2\|B\|^2 + \kappa(s))(C_2b + C_1nC_3\int_0^s E\|x(s)\|^2ds)ds \\ & \quad + 4N_2M(b + Tr(Q))(C_2b + C_1nC_3\int_0^t E\|x(s)\|^2ds), \\ & \leqslant 4N_2M|Fx(0)|^2 + 8N_2\frac{\widehat{\mathcal{N}}}{\lambda^2}(bM\|\rho F^{-1}\|^2\|B\|^2 + N_1)(C_2b + C_1nC_3b\|x\|^2) \\ & \quad + 4N_2M(b + Tr(Q))(C_2b + C_1nC_3b\|x\|^2) < \infty. \end{aligned}$$

Therefore, we obtain that $E\|(\Phi^{\lambda}x)(t)\|^2 < \infty$, that is, $\Phi^{\lambda}x(t) \in \mathcal{C}$, for every $x(t) \in \mathcal{C}$. Therefore, Φ^{λ} is self map. To apply the contraction mapping principle, now we prove under some condition, Φ^{λ} is a contraction on $\mathcal{C}$. To show this, let $x_1, x_2 \in \mathcal{C}$ then for $t \in [0, b]$, we have

$$\begin{aligned} & E\|(\Phi^{\lambda}x_1)(t) - (\Phi^{\lambda}x_2)(t)\|^2 \\ & = E\|\int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]B(u^{\lambda}(s,x_1) - u^{\lambda}(s,x_2))ds + \int_0^t F^{-1}T(t-s) \\ & \quad \times (f(s, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) - f(s, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s))))ds \\ & \quad + \int_0^t F^{-1}T(t-s)(g(s, x_1(\gamma_1(s)), x_1(\gamma_2(s)), \ldots, x_1(\gamma_n(s))) \\ & \quad - g(s, x_2(\gamma_1(s)), x_2(\gamma_2(s)), \ldots, x_2(\gamma_n(s))))dW(s)\|^2, \\ & \leqslant 3\frac{\mathcal{N}}{\lambda^2}N_2\int_0^t (M\|\rho F^{-1}\|^2\|B\|^2 + \kappa(s))\left(\int_0^t E\|x_1(s) - x_2(s)\|^2ds\right)ds + 3N_2M(b + Tr(Q))C_1nC_3 \\ & \quad \times \int_0^t E\|x_1(s) - x_2(s)\|^2ds, \\ & \leqslant 3\left(\frac{\mathcal{N}}{\lambda^2}N_2(bM\|\rho F^{-1}\|^2\|B\|^2 + N_1) + N_2M(b + Tr(Q))C_1nC_3\right)b\sup_{0 \leqslant t \leqslant b} E\|x_1(t) - x_2(t)\|^2, \end{aligned}$$

then it can be easily concluded that if (6) is satisfied, Φ^λ is a contraction map on a complete normed linear space $\mathcal{C}$, and thus by the contraction mapping theorem, has a unique fixed point in $\mathcal{C}$. □

3.2. Approximate boundary controllability

The following lemma gives a formula for a control steering the state x_0 to a neighborhood of $x_b \in L_2(\Omega, \mathfrak{F}_b, H)$.

Lemma 3.2. *For arbitrary $x_b \in L_2(\Omega, \mathfrak{F}_b, H)$, the control $u^\lambda(t,x)$ in (4) transfers the system (5) from x_0 to some neighborhood of x_b at time b and*

$$x^\lambda(b) = x_b - \lambda(\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1} \times (F^{-1}T(b-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s))) - \varphi(s))dW(s)$$

Proof. By substituting (4) in (5), one can easily obtain that

$$\begin{aligned}
x^\lambda(t) &= F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]B \\
&\quad \times \Big\{ B^\star F^{-1}[T(b-s)\rho F^{-1} - AF^{-1}T(b-s)]^\star (\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) \\
&\quad - B^\star F^{-1}[T(b-s)\rho F^{-1} - AF^{-1}T(b-s)]^\star \int_0^t (\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \\
&\quad \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds - B^\star F^{-1}[T(b-s)\rho F^{-1} - AF^{-1}T(b-s)]^\star \\
&\quad \times \int_0^t (\lambda I + \Gamma_s^b)^{-1}(F^{-1}T(b-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s))) - \varphi(s))dW(s) \Big\} ds \\
&\quad + \int_0^t F^{-1}T(t-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\
&\quad + \int_0^t F^{-1}T(t-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s), \\
&= F^{-1}T(t)Fx(0) + \int_0^t F^{-1}T(t-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\
&\quad + \int_0^t F^{-1}T(t-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s) + \Gamma_0^t T^\star(b-t)(\lambda I + \Gamma_0^b)^{-1} \\
&\quad \times (Ex_b - F^{-1}T(b)Fx(0)) - \int_0^t \Gamma_s^t T^\star(b-t)(\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \\
&\quad \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds - \int_0^t \Gamma_s^t T^\star(b-t)(\lambda I + \Gamma_s^b)^{-1}(F^{-1}T(b-s) \\
&\quad \times g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s))) - \varphi(s))dW(s)
\end{aligned}$$

The above equation can be rewritten at $t = b$, hence

$$\begin{aligned}
x^\lambda(b) - x_b &= F^{-1}T(b)Fx(0) + \int_0^b F^{-1}T(b-s)f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds \\
&\quad + \int_0^b F^{-1}T(b-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))dW(s) + (-\lambda I + \lambda I + \Gamma_0^b) \\
&\quad \times (\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) - \int_0^b (-\lambda I + \lambda I + \Gamma_s^b)(\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \\
&\quad \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds - \int_0^b (-\lambda I + \lambda I + \Gamma_s^b)(\lambda I + \Gamma_s^b)^{-1}(F^{-1}T(b-s) \\
&\quad \times g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s))) - \varphi(s))dW(s) - x_b, \\
x^\lambda(b) &= x_b - \lambda(\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \\
&\quad \times f(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s)))ds + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1} \\
&\quad \times (F^{-1}T(b-s)g(s, x(\gamma_1(s)), x(\gamma_2(s)), \ldots, x(\gamma_n(s))) - \varphi(s))dW(s). \quad \square
\end{aligned}$$

Theorem 3.3. *Assume the hypotheses (H_1)–(H_8) and Theorem 3.1 are satisfied. If f and g are uniformly bounded, then system (1) is approximately boundary controllable on J.*

Proof. By Theorem 3.1, Φ^λ has a unique fixed point x^λ in $\mathcal{C}$. By the stochastic Fubuni theorem and Lemma 3.2, it can be easily seen that

$$\begin{aligned}
x^\lambda(b) &= x_b - \lambda(\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1}F^{-1}T(b-s) \\
&\quad \times f(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s)))ds + \int_0^b \lambda(\lambda I + \Gamma_s^b)^{-1} \\
&\quad \times (F^{-1}T(b-s)g(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s))) - \varphi(s))dW(s)
\end{aligned}$$

It follows from the properties of f and g such that

$$\begin{aligned}
&\|f(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s)))\|^2 \\
&\quad + \|g(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s)))\|_Q^2 \\
&\quad \leqslant \mathcal{L}_1.
\end{aligned}$$

Then, there is a subsequence, still denoted by $\{f(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s))), g(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s)))\}$ which converges weakly to, say, $\{f(s,\upsilon), g(s,\upsilon)\}$ in $H \times L(K,H)$. On the other hand, by hypothesis (H_8), the operator $\lambda(\lambda I + \Gamma_s^b)^{-1} \to 0$ strongly as $\lambda \to 0^+$ and $\|\lambda(\lambda I + \Gamma_s^b)^{-1}\| \leqslant 1$ together with the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&E\|x^\lambda(b) - x_b\|^2 \\
&\leqslant 6E\|\lambda(\lambda I + \Gamma_0^b)^{-1}(Ex_b - F^{-1}T(b)Fx(0))\|^2 + 6\int_0^b \|\lambda(\lambda I + \Gamma_s^b)^{-1}\|^2 \|F^{-1}T(b-s)\|^2 \\
&\quad \times \|f(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s))) - f(s,\upsilon)\|^2 ds + 6\int_0^b \|\lambda(\lambda I + \Gamma_s^b)^{-1}\|^2 \|F^{-1}T(b-s)\|^2 \\
&\quad \times \|f(s,\upsilon)\|^2 ds + 6Tr(Q)\int_0^b \|\lambda(\lambda I + \Gamma_s^b)^{-1}\|^2 \|F^{-1}T(b-s)\|^2 \\
&\quad \times \|g(s, x^\lambda(\gamma_1(s)), x^\lambda(\gamma_2(s)), \ldots, x^\lambda(\gamma_n(s))) - g(s,\upsilon)\|_Q^2 ds + 6Tr(Q)\int_0^b \|\lambda(\lambda I + \Gamma_s^b)^{-1}\|^2 \\
&\quad \times \|F^{-1}T(b-s)\|^2 \|g(s,\upsilon)\|_Q^2 ds + 6Tr(Q)\int_0^b \|\lambda(\lambda I + \Gamma_s^b)^{-1}\|^2 \|\varphi(s)\|_Q^2 ds \to 0 \quad \text{as} \quad \lambda \to 0^+.
\end{aligned}$$

This gives the approximate boundary controllability of (1). □

Remark 3.4. Since many evolution processes, optimal control models in economics, stimulated neural networks, frequency modulated systems and some motions of missiles or aircrafts, automatic control systems, artificial intelligence, and robotics [37,38] are characterized by the dynamical systems with impulsive effects. However, in addition to impulsive effects, stochastic nature likewise exists in real systems. It is well known that a lot of dynamic systems have variable structures subject to stochastic perturbation, which may result from abrupt phenomena such as stochastic failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, etc. Therefore, the study of stochastic dynamical systems with impulsive effects is of great importance. Recently, the controllability problems for impulsive dynamical systems have been discussed in [7,17,18]. Thus, the obtained results in Theorem 3.3 can be extended to study the approximate boundary controllability of Sobolev-type stochastic differential systems with impulsive effects by employing the same idea and technique as discussed in Theorem 3.3.

4. Stochastic systems with poisson jumps

The stochastic model has come to play an important role in many branches of science and engineering. Such models have been used with great success in a variety of applications areas, including epidemiology, mechanics, economics, and finance. The modeling of risky asset by stochastic processes with continuous paths, based on Brownian motions, suffers from several defects. First, the path continuity assumption does not seem reasonable in view of the possibility of sudden price variations (jumps) resulting of market crashes. A solution is to use stochastic processes with jumps that will account for sudden variations of the asset prices. On the other hand, such jump models are generally based on the Poisson random measure. Many popular economic and financial models described by stochastic differential equations with Poisson jumps (see [39]). Ren et al. [40] discussed the existence, uniqueness, and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay under non-Lipschitz condition with Lipschitz condition being considered as a special case. Recently, Sakthivel et al. [10] studied the complete controllability of stochastic evolution equations with jumps without assuming the compactness of the semigroup property. In this section, we discuss boundary controllability for the stochastic differential systems with Poisson jumps in Hilbert spaces described in the form

$$\begin{aligned} d(Fx(t)) &= (\rho x(t) + f(t,x(t)))dt + g(t,x(t))dW(t) \\ &\quad + \int_{\mathcal{Z}} h(t,x(t),\eta)\widehat{N}(dt,d\eta), \qquad t \in J = [0,b], \\ \tau x(t) &= B_1 u(t), \\ x(0) &= x_0, \end{aligned} \tag{7}$$

where the functions $f{:}J\times H\to H$ and $g{:}J\times H\to L_Q(K,H)$. $\widehat{N}(ds,d\eta)$ is a compensated Poisson random measure induced by Poisson point process $k(\cdot)$, which is independent on the Wiener process W and takes values in a measurable space $(\mathcal{Z},\mathcal{B}(\mathcal{Z}))$ defined on a complete probability space $(\Omega,\mathfrak{F},P)$. $h: J\times H\times(\mathcal{Z}-\{0\})\to H$ be appropriate mappings. Further, let $\{k(t); t\in J\}$ be a Poisson point process which is independent of the Wiener process W, taking its values in a measurable space $(\mathcal{Z},\mathcal{B}(\mathcal{Z}))$ with a σ-finite intensity measure $\lambda_0(d\eta)$. We denote by $N(ds,d\eta)$ the Poisson counting measure, which is induced by $k(\cdot)$, and the compensating martingale measure by

$$\widehat{N}(ds,d\eta) = N(ds,d\eta) - \lambda_0(d\eta)ds.$$

It is to be assumed that the filtration generated by the Q-Wiener process $W(\cdot)$, the Poisson point process $k(\cdot)$ are augmented by,

$$\mathfrak{F}_t = \sigma\{W(s); s\leqslant t\}V\sigma\{N((0,s),A); s\leqslant t, A\in\mathcal{B}(\mathcal{Z})\}VN_0, \quad t\in J,$$

where N_0 is the class of P-null sets.

Similar to Eq. (3) in Section 2, the mild solution of the system (7) is given by

$$\begin{aligned} x(t) &= F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu(s)ds \\ &\quad + \int_0^t F^{-1}T(t-s)f(s,x(s))ds + \int_0^t F^{-1}T(t-s)g(s,x(s))dW(s) \\ &\quad + \int_0^t\int_{\mathcal{Z}} F^{-1}T(t-s)h(s,x(s),\eta)\widehat{N}(ds,d\eta). \end{aligned} \tag{8}$$

(H_9) The functions f and g are continuous and there exist constants C_4, C_5, for $t\in J$ and x_1, $x_2\in H$, such that

$$\|f(t,x_1)-f(t,x_2)\|^2 + \|g(t,x_1)-g(t,x_2)\|_Q^2 \leqslant C_4\|x_1-x_2\|^2,$$
$$C_5 = \max_{t\in J}(\|f(t,0)\|^2 + \|g(t,0)\|^2)$$

(H_{10}) The nonlinear function h is continuous and there exist constants C_6, C_7, C_8, C_9, for $t\in J$ and x_1, $x_2\in H$, such that

$$\int_{\mathcal{Z}} \|h(t,x_1,\eta)-h(t,x_2,\eta)\|^2\lambda(d\eta) \leqslant C_6\|x_1-x_2\|^2,$$
$$\int_{\mathcal{Z}} \|h(t,x_1,\eta)-h(t,x_2,\eta)\|^4\lambda(d\eta) \leqslant C_7\|x_1-x_2\|^4,$$
$$\int_{\mathcal{Z}} \|h(t,x,\eta)\|^2\lambda(d\eta) \leqslant C_8(1+\|x\|^2),$$
$$\int_{\mathcal{Z}} \|h(t,x,\eta)\|^4\lambda(d\eta) \leqslant C_9(1+\|x\|^4).$$

Clearly, under the hypotheses (H_9)–(H_{10}), for every $u(\cdot)\in L_2^{\mathfrak{F}}(J,U)$, the integral Eq. (8) has a unique solution in $\mathcal{C}$. To apply the contraction mapping principle, we define the nonlinear operator Φ_1^λ from $\mathcal{C}$ into itself as follows

$$\begin{aligned} (\Phi_1^\lambda x)(t) &= F^{-1}T(t)Fx(0) + \int_0^t F^{-1}[T(t-s)\rho F^{-1} - AF^{-1}T(t-s)]Bu^\lambda(s,x)ds \\ &\quad + \int_0^t F^{-1}T(t-s)f(s,x(s))ds + \int_0^t F^{-1}T(t-s)g(s,x(s))dW(s) \\ &\quad + \int_0^t\int_{\mathcal{Z}} F^{-1}T(t-s)h(s,x(s),\eta)\widehat{N}(ds,d\eta), \end{aligned} \tag{9}$$

where

$$\begin{aligned} u^\lambda(t,x) &= B^\star F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^\star\left(\lambda I+\Gamma_0^b\right)^{-1}(Ex_b - F^{-1}T(b)Fx(0)) \\ &\quad - B^\star F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^\star\int_0^t\left(\lambda I+\Gamma_s^b\right)^{-1}F^{-1}T(b-s)f(s,x(s))ds \\ &\quad - B^\star F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^\star\int_0^t\left(\lambda I+\Gamma_s^b\right)^{-1}F^{-1}T(b-s)g(s,x(s))dW(s) \\ &\quad - B^\star F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^\star\int_0^t\int_{\mathcal{Z}}(\lambda I+\Gamma_s^b)^{-1}F^{-1}T(b-s)h(s,x(s),\eta) \\ &\quad \times\widehat{N}(ds,d\eta) + B^\star F^{-1}[T(b-t)\rho F^{-1} - AF^{-1}T(b-t)]^\star\int_0^t\left(\lambda I+\Gamma_s^b\right)^{-1}\varphi(s)dW(s). \end{aligned}$$

Theorem 4.1. *Assume that the hypotheses (H_1)–(H_5) and (H_8)–(H_{10}) hold. Then, the system (7) is approximately boundary controllable on [0,b] provided*

$$4\left(\mathcal{N}_1\frac{N_2}{\lambda^2}(bM\|\rho F^{-1}\|^2\|B\|^2+N_1)+N_2MC_4(b+Tr(Q))+N_2Mb\left(C_6+\sqrt{C_7}\right)\right)b < 1.$$

Proof. The proof of this theorem is similar to that of Theorems 3.1 and 3.3 and one can easily prove that if for all $\lambda > 0$, the operator Φ_1^λ has a fixed point by employing the contraction mapping principle used in the Theorem 3.1, then we can show that the system (7) is approximately boundary controllable (similar Theorem 3.3) and hence it is omitted. □

5. Example

Consider the following system of nonlinear stochastic partial differential equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t}(z(t,y)-\Delta z(t,y)) = \Delta z(t,y) + f(t,z(\alpha_1(t),y),z(\alpha_2(t),y),\ldots,z(\alpha_n(t),y)) \\ &\quad + g(t,z(\alpha_1(t),y),z(\alpha_2(t),y),\ldots,z(\alpha_n(t),y))\partial\beta(t), \quad t\in J=[0,b], \quad y\in\Lambda, \\ &z(t,y) = u(t), \quad t\in J, \quad y\in\xi, \\ &z(0,y) = z_0(y), \quad y\in\Lambda, \end{aligned} \tag{10}$$

where Λ is a bounded and open subset of $\mathbb{R}^n$ with a sufficiently smooth boundary ξ. Let $H = L_2(\Lambda)$, $\beta(t)$ denotes a one dimensional standard Brownian motion in H defined on a stochastic space $(\Omega,\mathfrak{F},P)$. The above problem can be formulated abstractly into the boundary control system (1) by suitably choosing $U = L_2(\xi)$, $Y = Z = L_2(\Lambda)$, $B_1 = I$, the operator $F{:}D(F) \subset Y \to Z$ defined by $Fw = w - \Delta w$ with $D(F) = H^2(\Lambda)$ and $D(\rho) = \{z \in L_2(\Lambda);\ \Delta z \in L_2(\Lambda)\}$, $\rho z = \Delta z$. The operator τ is the trace operator $\tau z = z|_{\xi}$ is well defined and belongs to $H^{-\frac{1}{2}}(\xi)$ for each $z \in D(\rho)$ (see [28]). Take $\alpha_i(t) = k_i t$, $t \in J$, $k_i \in (0,1]$ for $i = 1, 2, \dots, n$. Observe that $\alpha_i{:}J \to J$ is a bounded continuous function. Define the operator $A{:}D(A) \subset Y \to Z$ by $AF^{-1}w = \Delta F^{-1}w$ with $D(AF^{-1}) = H_0^1(\Lambda) \cup H^2(\Lambda)$. Then, A and F can be written respectively as

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

$$Fw = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n) w_n, \quad w \in D(F),$$

where $w_n(y) = \sqrt{2}\sin ny$, $n = 1, 2, 3, \dots$ is the orthogonal set of eigenvectors of A. Furthermore, for $w \in Y$

$$F^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(w, w_n) w_n,$$

$$AF^{-1}w = \sum_{n=1}^{\infty} \frac{n^2}{1+n^2}(w, w_n) w_n,$$

$$T(t)w = \sum_{n=1}^{\infty} e^{\frac{n^2 t}{1+n^2}}(w, w_n) w_n.$$

It is easy to see that AF^{-1} generates a strongly continuous semigroup $T(t)$ on Z. Hence, the hypotheses (H_1), (H_2) are satisfied. Define the linear operator $B{:}L_2(\xi) \to L_2(\Lambda)$ by $Bu = v_u$, where $v_u \in L_2(\Lambda)$ is the unique solution to the Dirichlet boundary value problem

$$\begin{aligned} \Delta v_u &= 0 \quad \text{in} \quad \Lambda, \\ v_u &= u \quad \text{in} \quad \xi. \end{aligned} \tag{11}$$

It is proved in [23] that for every $u \in H^{-\frac{1}{2}}(\xi)$, the Eq. (11) has a unique solution $v_u \in L_2(\Lambda)$ satisfying $\|Bu\|_{L_2(\Lambda)} = \|v_u\|_{L_2(\Lambda)} = c_1 \|u\|_{H^{-\frac{1}{2}}(\xi)}$. This shows that (H_3) is satisfied. From the above estimates, it follows by an interpolation argument [30] that $\|AF^{-1}T(t)B\|_{\mathcal{L}(L_2(\xi), L_2(\xi))} \leqslant c_2 t^{-\frac{3}{4}}$, for all $t > 0$ with $v(t) = c_2 t^{-\frac{3}{4}}$, where c_1,c_2 are positive constants independent of u. Therefore, the hypotheses (H_4), (H_5) are satisfied. The approximate boundary controllability of the corresponding linear system of (10) is discussed in great detailed in [24,41]. Clearly, the nonlinear functions f and g satisfies the hypotheses (H_6), (H_7). All conditions stated in the Theorem 3.3 are satisfied; therefore, the system (10) is approximately boundary controllable on J.

6. Conclusion

In this paper, the boundary controllability results for Sobolev-type stochastic differential system is discussed. The existence and uniqueness results of the mild solution of Sobolev-type stochastic differential system are obtained by using the Banach fixed point theorem. The sufficient conditions for approximate controllability of this system is proved under natural assumption that the corresponding linear system is approximately controllable. The derived result shows that how the Banach fixed point theorem can effectively be used in control problems. In addition, the boundary controllability results of stochastic differential systems with Poisson jumps is proved. The effectiveness of the theoretical results is finally verified with suitable stochastic partial differential equations. In future, the authors interested to study the boundary controllability of fractional order Sobolev-type stochastic integro differential systems by using Poisson random measures and multiplicative Levy noises.

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